Lecture 14

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1 Functions

In previous lectures we worked with algebraic structures — sets with operations defined on them. Now we will consider another important thing in mathematics — functions.

Let A and B be 2 sets. **Function** f from A to B can be considered as a rule, which allows us to get an element from B for any element from A. The notation for a function from the set A to the set B is: $f : A \to B$. Set A is called the **domain** of a function f. We will often use the following notation: $x \mapsto f(x)$, which denotes that x maps to $f(x)$, i.e. applying f to x we get $f(x)$.

Now let's consider any element x from A. Then $f(x) \in B$ is called the **image** of x. Moreover we can consider the subset $A' \subset A$. Then by $f(A')$ we will denote the set which contains images of all the elements from A' and it will be called the **image** of A'.

Let's consider any subset in B, say, $B' \in B$. Then by $f^{-1}(B')$ we will denote all elements from A, whose images are in B'. $f^{-1}(B')$ will be called the **inverse image** of **preimage** of B'.

Example 1.1. *Consider the function $f(x) = x^2$. This function is defined for any real number, and maps them to nonnegative real numbers. If $\mathbb{R}_+$ denotes positive numbers, then*

$$f : \mathbb{R} \to \mathbb{R}_+ \cup \{0\},$$

where $\mathbb{R}_+ \cup \{0\}$ is the set of all nonnegative numbers.

Image of the set $[-2, 2]$ is the set $[0, 4] = f([-2, 2])$.

Inverse image of the set $[4, 25]$ is the set $[-5, -2] \cup [2, 5] = f^{-1}([4, 25])$.

Consider another important notion — composition of functions. Let $f : A \to B$, $g : B \to C$ are 2 functions. Then **composition** of f and g is the function which is denoted by $g \circ f$ such that $g \circ f : A \to C$, and

$$(g \circ f)(x) = g(f(x)).$$

We state here the obvious property of compositions: if $f : A \to B$, $g : B \to C$, and $h : C \to D$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Example 1.2. *Consider* $f : \mathbb{R}^2 \to \mathbb{R}$ *such that* $(x, y) \mapsto xy$*, and* $g : \mathbb{R} \to \mathbb{R}$ *such that* $x \mapsto x^3$*. Then* $g \circ f : \mathbb{R}^2 \to \mathbb{R}$ *and* $(g \circ f)(x, y) = g(f(x, y)) = g(xy) = (xy)^3$*.*

Now, let A be a nonempty set. The function $f : A \to A$ such that for any $x \in A$ we have $f(x) = x$ is called the **identity function**. It will be denoted by I: $I(x) = x$.

Now let $f : A \to B$. Function $g : B \to A$ is called the **inverse function** for f if

$$(f \circ g)(x) = x \ \forall x \in B, \text{ and } (g \circ f)(x) = x \ \forall x \in A.$$

Example 1.3. *Let* $f : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ *such that* $f(x) = x^2$*. Then* $g(x) = \sqrt{x}$ *is not an inverse: let's take* $x = -2$*, then* $-2 \overset{f}{\longmapsto} 4 \overset{g}{\longmapsto} 2$*.*

But if we consider $f(x) = x^2$ *only for nonnegative numbers, then* $g(x) = \sqrt{x}$ *will be the inverse.*

All these definitions are basic definitions of mathematics, and are not specific for linear algebra.

2 Linear functions

Now, we will consider a class of functions, which is specific to linear algebra.

Let V and U are vector spaces.

Definition 2.1. *Function* $f : V \to U$ *is called a* ***linear function*** *if the following 2 conditions are satisfied:*

- *For any vectors* v *and* w *from* V

$$f(v + w) = f(v) + f(w)$$

- *For any vector* $v \in V$ *and for any number* $k \in \mathbb{R}$

$$f(kv) = kf(v)$$

Now we'll give some examples of linear functions.

Example 2.2. *The identity function is a linear function. It is easy to see:*

$$I(x + y) = x + y = I(x) + I(y)$$
$$I(kx) = kx = kI(x)$$

Example 2.3. *Let's consider the zero-function — function* $f : V \to V$ *such that for any* $v \in V$ $f(v) = 0$*. This function is obviously a linear function.*

Example 2.4 (Projection). *Let's consider the space* $\mathbb{R}^3$*, and let* f *be a function such that* $f : \mathbb{R}^3 \to \mathbb{R}^3$*, and*

$$(x, y, z) \xmapsto{f} (x, y, 0)$$

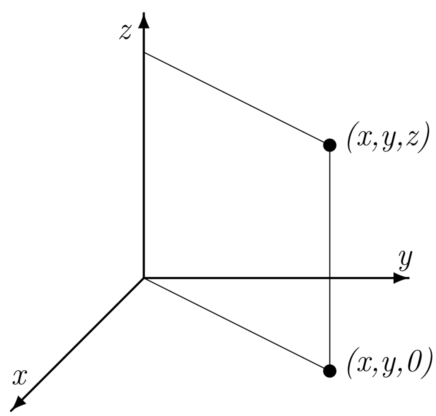
Then this is a linear function. Let's check the first property:

$$f(x_1, y_1, z_1) = (x_1, y_1, 0); \quad f(x_2, y_2, z_2) = (x_2, y_2, 0)$$
$$f(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (x_1 + x_2, y_1 + y_2, 0) = f(x_1, y_1, z_1) + f(x_2, y_2, z_2).$$

And the second property:

$$f(kx, ky, kz) = (kx, ky, 0) = k(x, y, 0) = kf(x, y, z).$$

This function is called the projection. We can consider triplets of numbers as points in the 3-dimensional space, and then this function maps any point to its projection to the xy-plane.



Example 2.5. *Consider the vector space of all polynomials* $\mathbb{P}$*. We'll give this definition which is familiar from the course of calculus.*

Definition 2.6. *The* ***derivative of a polynomial***

$$P(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_i t^i + \cdots + a_1 t + a_0$$

is a polynomial

$$DP(t) = P'(t) = n a_n t^{n-1} + (n-1) a_{n-1} t^{n-2} + \cdots + i a_i t^{i-1} + \cdots + a_1.$$

Let the function D *is such that* $D : \mathbb{P} \to \mathbb{P}$*, and*

$$D(p(t)) = p'(t),$$

i.e. the given polynomial maps to its derivative. This function is linear by basic properties of derivative:

$$(f + g)' = f' + g' \text{ and}$$
$$(kf)' = kf'$$

Now we'll state very easy result about linear functions.

Lemma 2.7. *If f is a linear function, then $f(\mathbf{0}) = \mathbf{0}$.*

Proof. Let $k \neq 0$. Then $f(k\mathbf{0}) = kf(\mathbf{0})$. Moreover, since $k\mathbf{0} = \mathbf{0}$, then $f(k\mathbf{0}) = f(\mathbf{0})$. So, comparing these two equalities, we have that $f(\mathbf{0}) = kf(\mathbf{0})$, so $f(\mathbf{0}) = \mathbf{0}$. □

Example 2.8. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x, y) = (x + 1, y + 2)$. Then this is not linear function, since the image of zero is not zero:*

$$f(0, 0) = (1, 2) \neq (0, 0).$$

Actually, all other properties do not hold here as well. For example, let $u = (1, 1)$, and let $v = (1, 0)$. Then $f(u) = (2, 3)$, $f(v) = (2, 2)$, and so, $f(u) + f(v) = (4, 5)$. But $f(u + v) = f(2, 1) = (3, 3) \neq (4, 5)$.

Now we will consider the most important linear function which will be used widely in the future.

Example 2.9 (Matrix function). *Let A be any $m \times n$-matrix. Then we can define the function $F_A : \mathbb{R}^n \to \mathbb{R}^m$ by the following formula: if $v \in \mathbb{R}^n$ then $v \xmapsto{F_A} Av$, where Av is a multiplication of a matrix A by a column-vector ($n \times 1$-matrix) v.*

Consider an example. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}$, so $F_A : \mathbb{R}^3 \to \mathbb{R}^2$. If $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then

$$F_A(v) = Av = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}.$$

So, the image of vector $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is vector $Av = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$.

This function is obviously linear:

$$F_A(u + v) = A(u + v) = Au + Av = F_A(u) + F_A(v),$$
$$F_A(ku) = A(ku) = k \cdot Au = kF_A(u).$$